COLORING BLOCKS OF CONSECUTIVE INTEGERS TO FORBID THREE DISTANCES

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1 Introduction

Let a, b, and c be positive integers such that $a < b < c$; suppose we want to color the set $S_n = \{1, 2, ..., n\}$ with three colors such that the distances a, b, and c are *forbidden*, meaning that if $|x - y| \in \{a, b, c\}$ for $x, y \in S_n$, then x and y must be colored differently. What is the largest n such that this is possible? It turns out that this question is most interesting when $a + b = c$. Before we further discuss the case $c = a + b$, we make some definitions.

Definition Let a, b, and c be positive integers such that $a < b < c$. The minimum number of colors needed to color a set of integers S such that the distances a, b, and c are forbidden is the *chromatic number* of S with respect to the forbidden distances a, b, and c, and is denoted by $\chi(S, \{a, b, c\})$.

Chen, Chang, and Huang [1] proved the following theorem:

Theorem 1. *Let a, b, and c be positive integers such that $a < b < c$, $\gcd(a, b, c) = 1$, and at least one of $\{a, b, c\}$ is even. Then $\chi(\mathbb{Z}, \{a, b, c\}) \in \{3, 4\}$. $\chi(\mathbb{Z}, \{a, b, c\}) = 3$ if one of the following is true:*

1. $c = a + b$ and $a \equiv b \mod 3$.
2. $a = 1$, $b = 2$, and $c \not\equiv 0 \mod 3$.
3. $a \geq 2$, $b = a + 1$, and $c \neq 2a + 1$.

Furthermore, $\chi(\mathbb{Z}, \{a, b, c\}) = 4$ if one of the following is true:

1. $c = a + b$ and $a \not\equiv b \mod 3$.
2. $a = 1$, $b = 2$, and $c \equiv 0 \mod 3$.

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2 Maximum Lengths

We are interested in the cases wherein the chromatic number of the integers with respect to the forbidden distances $\{a, b, c\}$ is 4 because it is in those cases that the largest n mentioned in the first sentence of this paper exists.

Definition Let $S_n = \{1, 2, ..., n\}$ be the set of all positive integers less than or equal to n. Given $\{a, b, c\}$ such that $a < b < c$ and $\chi(\mathbb{Z}, \{a, b, c\}) = 4$, we say that the maximum n such that $S_n = \{1, 2, ..., n\}$ can be three colored so that the distances a, b, and c are forbidden is the *maximum length* with respect to $\{a, b, c\}$, and is denoted

$$\max(n : \chi(S_n, \{a, b, c\}) < 4) = \mathrm{ML}(a, b, c) \tag{1}$$

If $\chi(\mathbb{Z}, \{a, b, c\}) < 4$, we say that $\mathrm{ML}(a, b, c) = \infty$.

Example 1. What is $\mathrm{ML}(1, 2, 3)$? In $S_n = \{1, 2, ..., n\}$, let's start by coloring the number 1 red. Then the number 2 must be different since it is a distance 1 from the number 1; let's color it yellow. Finally, the number 3, being a distance 1 from the number 2 and a distance 2 from the number 1, must be some other color; why not blue? But now we are in trouble, since there are no colors left for the number 4. Therefore, $\mathrm{ML}(1, 2, 3) = 3$.

Example 2. What is $\mathrm{ML}(1, 2, n)$ when $n \not\equiv 0 \mod 3$? We may color the integers like so:

$$C(i) = \begin{cases} \mathit{Red} & : i \equiv 0 \mod 3 \\ \mathit{Yellow} & : i \equiv 1 \mod 3 \\ \mathit{Blue} & : i \equiv 2 \mod 3 \end{cases}$$

Clearly this coloring respects the forbidden distances $1, 2$, and n. Therefore, when $n \not\equiv 0 \mod 3$, we have $\chi(\mathbb{Z}, \{1, 2, n\}) \leq 3$ and so $\mathrm{ML}(1, 2, n) = \infty$. (Note: we could also have gotten $\chi(\mathbb{Z}, \{1, 2, n\}) = 3$ from Theorem 1; in fact, the coloring given here proves one claim of Theorem 1).

Theorem 2. If $n \geq 3$ and $n \equiv 0 \mod 3$, then $\mathrm{ML}(1, 2, n) = n$

Proof. With the forbidden distances 1 and 2, we are forced into a coloring as in Example 2 wherein a period of 3 different colors (for example, $\{\mathit{Red}, \mathit{Yellow}, \mathit{Blue}\}$) is repeated. This suffices to color S_n, whence we have $\mathrm{ML}(1, 2, n) \geq n$. But in an attempt to 3-color S_{n+1}, the above coloring would result in 1 and $n+1$ being the same color, since $n+1 \equiv 1 \mod 3$. Thus, $\mathrm{ML}(1, 2, n) \equiv n$ when $n \equiv 0 \mod 3$. □

With that settled, we move on to the case that $c = a + b$. We start with a useful lemma, for which we are indebted to [1]:

Lemma 3. Let S_n be colored with three colors so that distances a, b, and $a+b$ are forbidden.

1. For all $x \in S_n$, if $a + 1 \leq x \leq n - b$, then x and $x + b - a$ are the same color.

2. For all $x \in S_n$, if $x \leq n - (2a + b)$, then x and $x + 2a + b$ are the same color.

Proof. For 1 we note that $x - a$, x, and $x + b$ have different colors, and that $x - a$, $x + b$, and $x + b - a$ have different colors. Since there are only three colors, x and $x + b - a$ must be the same color.
For 2 we note that x, $x + a$, and $x + a + b$ have different colors, and that $x + a$, $x + a + b$, and $x + 2a + b$ have different colors. Since there are only 3 colors, x and $x + 2a + b$ are the same color. □

We now prove several formulae for $\mathrm{ML}(a, b, a + b)$ for certain a, b.

Theorem 4.

$$\mathrm{ML}(1, b, b+1) = \begin{cases} 2b & : b \equiv 0 \mod 3 \\ \infty & : b \equiv 1 \mod 3 \\ 2b - 1 & : b \equiv 2 \mod 3 \end{cases}$$

Proof. When $b \equiv 1 \mod 3$, $\chi(\mathbb{Z}, \{1, b, b+1\}) = 3$ by Theorem 1, so $\mathrm{ML}(1, b, b+1) = \infty$ in that case.
For the other cases, supposing that S_n is colored with red, yellow, and blue so that distances $1, b$, and $b + 1$ are forbidden, we note that if $x \leq n - b - 2$, then x and $x + 2$ must be different colors. To see this, we suppose otherwise that x and $x + 2$ were both the same color, say red. Clearly $x + 1$ must be a different color, so let it be yellow. Now $x + (b + 1)$ must be different from x and $x + 1$, so it must be blue. But now $x + (b + 2)$ cannot be colored, since $x + 1$ is yellow, $x + 2$ is red, and $x + (b + 1)$ is blue.
Combining the above observation with the fact that 1 is a forbidden distance, we see that the first $n - b$ numbers of S_n must be colored based on their value modulo 3, as in Example 2. Furthermore, for $b + 1 \leq x \leq n - 1$, we must have x colored the same as $x - b + 1$, since it must be different from $x - b$ and $x - b - 1$ (when $x = b + 1$, its color must be different from those of 1 and $b + 2$, which both must be different from the color of 2).
Now suppose $b \equiv 2 \mod 3$ and $n = 2b$. We start with 1 colored red and 2 colored yellow. The forced coloring just described results in $b \equiv 2 \mod 3$ colored yellow and $b + 1$ colored the same as $b + 1 - b + 1 = 2$, which is also yellow. This is a contradiction, whence $\mathrm{ML}(1, b, b + 1) < 2b$ when $b \equiv 2 \mod 3$. On the other hand, if $n = 2b - 1$, then we have the numbers 1 through $b - 1$ colored

$$R, Y, B, ..., R$$

and the numbers $b + 1$ through $2b - 2$ colored

$$Y, B, R, ..., R$$

This coloring has all x colored differently from $x + 1$, $x + b$, and $x + b + 1$, when these are colored. In this case, we may safely color the number b blue and the number $2b - 1$ yellow, whereupon we conclude that $\mathrm{ML}(1, b, b+1) = 2b - 1$ when

$b \equiv 2 \mod 3$.
Finally, suppose $b \equiv 0 \mod 3$ and $n = 2b+1$. We start again with 1 colored red and 2 colored yellow. What color is $b+1$? Since $b+1 \leq n-b$, it should be colored red based on its value modulo 3. However, it also needs to be the same color as $b+1-b+1 = 2$, which is yellow. This is a contradiction, whence $\mathrm{ML}(1, b, b+1) < 2b+1$ when $b \equiv 0 \mod 3$. On the other hand, if $n = 2b$, then we have the numbers 1 through b colored

$$R, Y, B, ..., B$$

and the numbers $b+1$ through $2b-1$ colored

$$Y, B, R, ..., B$$

This coloring has all x colored differently from $x+1$, $x+b$, and $x+b+1$, when these are colored. In this case, we may safely color the number $2b$ red, whereupon we conclude that $\mathrm{ML}(1, b, b+1) = 2b$ when $b \equiv 0 \mod 3$. □

Theorem 5.

$$\mathrm{ML}(a, a+1, 2a+1) = 3a$$

Proof. In S_{3a}, the following 3-coloring respects the forbidden distances.

- If $i \leq a$, color i red.
- If $a < i \leq 2a$, color i yellow.
- If $i > 2a$, color i blue.

Therefore $\mathrm{ML}(a, a+1, 2a+1) \geq 3a$. Now toward contradiction, assume $\mathrm{ML}(a, a+1, 2a+1) > 3a$. We may as well assume that $a > 1$. Then S_{3a+1} can be 3-colored with respect to the forbidden distances. We may as well start by coloring the number 1 red, $a+1$ yellow, and $2a+2$ blue. By Lemma 3, we are now forced to make $a+2$ yellow. Here's what we have so far:

Number	1	...	$a+1$	$a+2$	...	$2a+2$	...	$3a+1$
Color	R	?	Y	Y	?	B	?	?

Applying Lemma 3 with $x = a+2$, we must have $a+3$ yellow. Then, applying Lemma 3 with $x = a+3$, we must have $a+4$ yellow. This process will conclude with $2a+1$ colored yellow. But then $2a+1$ and $a+1$ are both yellow, and we're in trouble. Therefore, $\mathrm{ML}(a, a+1, 2a+1) = 3a$. □

It turns out that our results are not only discrete; with the following theorem, we make our results continuous:

Theorem 6. Let a, b, and c be integer forbidden distances. S_n can be 3-colored with respect to the forbidden distances if and only if the real interval $[0, n)$ can be as well.

Proof. Assume S_n can be 3-colored with respect to the forbidden distances (this is equivalent to $\mathrm{ML}(a,b,c) \geq n$). The following is a valid 3-coloring of the real interval $[0,n)$ with respect to the forbidden distances:

- The color of $x \in [0,n)$ is the same as the color of $(\lfloor x \rfloor + 1) \in S_n$.

Now assume the real interval $[0,n)$ can be 3-colored with respect to the forbidden distances. The following is a valid 3-coloring of S_n with respect to the forbidden distances:

- The color of $x \in S_n$ is the same as the color of $(x-1) \in [0,n)$.

□

This leads us to a theorem which can reduce any problem with forbidden distances to a problem with relatively prime forbidden distances.

Theorem 7. For any positive integer k and integer forbidden distances a, b, c,

$$\mathrm{ML}(ka, kb, kc) = k \cdot \mathrm{ML}(a,b,c)$$

Proof. We begin by showing that $\mathrm{ML}(ka,kb,kc) \geq k \cdot \mathrm{ML}(a,b,c)$. Let $\mathrm{ML}(a,b,c) = n$. Then there is a 3-coloring of S_n with respect to the forbidden distances a, b, and c. By Theorem 6, this gives us a 3-coloring of the real interval $[0,n)$ with respect to the forbidden distances a, b, and c. Let this coloring be $C(x)$, a map from $[0,n)$ to {red, yellow, blue}. Now color the interval $[0,kn)$ like so: the color of $x \in [0,kn)$ is the color given by $C(x/k)$. This results in a "stretched" version of C, a 3-coloring of $[0,kn)$ which respects the forbidden distances ka, kb, and kc. By Theorem 6, the existence of this coloring implies that $\mathrm{ML}(ka,kb,kc) \geq kn$, or $\mathrm{ML}(ka,kb,kc) \geq k \cdot \mathrm{ML}(a,b,c)$.

Now, toward contradiction, suppose $\mathrm{ML}(ka,kb,kc) > k \cdot \mathrm{ML}(a,b,c)$. Equivalently, suppose $\mathrm{ML}(ka,kb,kc) \geq k \cdot \mathrm{ML}(a,b,c) + 1$. Let $\mathrm{ML}(a,b,c) = n$ again. Then there is a 3-coloring of S_{kn+1} with respect to the forbidden distances ka, kb, and kc. By Theorem 6, this gives us a 3-coloring of the real interval $[0, kn+1)$ which respects the forbidden distances ka, kb, and kc. Let this coloring be $D(x)$, a map from $[0,kn+1)$ to {red, yellow, blue}. Now color the interval $[0, n+\frac{1}{k})$ like so: the color of $x \in [0, n+\frac{1}{k})$ is given by $D(kx)$. This results in a "shrunken" version of D, a 3-coloring of $[0, n+\frac{1}{k})$ which respects the forbidden distances a, b, and c. Let this coloring of $[0, n+\frac{1}{k})$ be $D'(x) = D(kx)$, a map from $[0, n+\frac{1}{k})$ to {red, yellow, blue}. Now color S_{n+1} like so: the color of $i \in S_{n+1}$ is given by $D'(i-1)$ (we subtract 1 because the real interval starts at 0 and S_{n+1} starts at 1). Now we have a 3-coloring of S_{n+1} which respects the forbidden distances a, b, and c. Then it must be the case that $\mathrm{ML}(a,b,c) \geq n+1$. But we started with $\mathrm{ML}(a,b,c) = n$, so we have arrived at a contradiction. Thus, it must be the case that $\mathrm{ML}(ka,kb,kc) = k \cdot \mathrm{ML}(a,b,c)$. □

Corollary 8. Suppose a, b, and c are forbidden distances with common divisor d. Then

$$\mathrm{ML}(a,b,c) = d \cdot \mathrm{ML}\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$$

The proof is trivial.

Corollary 9. When a is even, $\mathrm{ML}(a, a+2, 2a+2) = 3a$.

Proof. By Corollary 8,

$$\mathrm{ML}(a, a+2, 2a+2) = 2 \cdot \mathrm{ML}\left(\frac{a}{2}, \frac{a}{2}+1, a+1\right)$$

By Theorem 5, we get

$$2 \cdot \mathrm{ML}\left(\frac{a}{2}, \frac{a}{2}+1, a+1\right) = 2 \cdot \left(3 \cdot \frac{a}{2}\right) = 3a$$

□

Before attempting $\mathrm{ML}(a, a+2, 2a+2)$, we prove a theorem which will help us avoid exhausting searches for forced colorings. The author is indebted to Pete Johnson for this theorem:

Theorem 10. Suppose that a and b are positive integers, $a < b$, $\lambda, \mu \in \mathbf{N}$ and

$$1 + \mu(2a+b) = a + 1 + \lambda(b-a) \tag{2}$$

Then if S_n is three colored with respect to forbidden distances $a, b, a+b$, then

$$n - b \leq a + (\lambda - 1)(b-a)$$

or, equivalently,

$$\mathrm{ML}(a, b, a+b) \leq 2a + \lambda(b-a)$$

Proof. Note that $\mu, \lambda \in \mathbf{N}$, $a, b > 0$, $a < b$, and

$$\mu(2a+b) = a + \lambda(b-a)$$

Thus, we must have $\mu \geq 1$ and $\lambda \geq 2$. Now suppose we have μ and λ which satisfy equation (2). Assume towards contradiction that S_n can be 3-colored with respect to forbidden distances a, b, and $a+b$, and where

$$n \geq 2a + 1 + \lambda(b-a)$$

For each $t \in \{0, ..., \lambda - 1\}$, we have

$$a + 1 \leq a + 1 + t(b-a) \leq 2a + 1 + \lambda(b-a) - b \leq n - b$$

By Lemma 3, with $x = a+1+t(b-a)$ for $0 \leq t \leq \lambda - 1$, x and $x+b-a$ have the same color. Therefore, $a+1$, $a+1+b-a$, $a+1+2(b-a)$, ..., $a+1+\lambda(b-a)$ all have the same color.
By equation (2), we have

$$1 \leq a + 1 + \lambda(b-a) = 1 + \mu(2a+b) \leq n - a < n$$

We will show that $a+1+\lambda(b-a)$ must have the same color as 1, with an argument that will be given shortly; if this is so, then 1, $a+1$, and $a+1+\lambda(b-a)$ must be the same color. But $(a+1)-a=1$, so that can't be possible.
Now how do we see that 1 and $1+\mu(2a+b)$, which is $\le n-a < n$, are the same color? The argument is similar to the one that established that $a+1$ and $a+1+\lambda(b-a)$ are the same color.
For $0 \le t \le \mu - 1$,

$$1 \le 1 + t(2a+b) \le 1 + (\mu-1)(2a+b) = 1 + \mu(2a+b) - (2a+b)$$

$$\le n - (2a+b)$$

Therefore, by Lemma 3, for each $t \in [0, ..., \mu-1]$, $1+t(2a+b)$ and $1+(t+1)(2a+b)$ have the same color. Thus, 1, $1+2a+b$, $1+2(2a+b)$, ..., $1+\mu(2a+b)$ all have the same color. Now 1 and $a+1$ have the same color, and we've arrived at a contradiction. $\square$

Theorem 11.

$$\mathrm{ML}(a, a+2, 2a+2) = \begin{cases} 3a & : a \equiv 0 \mod 2 \\ 4a+2 & : a \equiv 1 \mod 2 \end{cases}$$

Proof. Corollary 9 gives us $\mathrm{ML}(a, a+2, 2a+2) = 3a$ when a is even - now we concern ourselves with when a is odd. The following is a valid 3-coloring of S_{4a+2} when a is odd.
Take $x \in S_{4a+2}$.

- If $x \le a$ or $x > 3a+2$, color x red.
- If $a < x \le 3a+2$ and x is even, color x yellow.
- If $a < x \le 3a+2$ and x is odd, color x blue.

Thus $\mathrm{ML}(a, a+2, 2a+2) \ge 4a+2$ when a is odd. Towards application of Theorem 10, let $\mu = 1$ and $\lambda = a+1$. Then, equation (2) becomes

$$1 + (2a + a + 2) = a + 1 + 2(a+1)$$

This equation is true. Thus, by Theorem 10, $\mathrm{ML}(a, a+2, 2a+2) \le 4a+2$. This proves the theorem. $\square$

Theorem 12. Given n a positive integer, if $a \equiv 0 \mod n$ then

$$\mathrm{ML}(a, a+n, 2a+n) = 3a$$

Proof. Since $a \equiv 0 \mod n$, we have $\gcd(a, a+n, 2a+n) = n$. Then, by Corollary 8, we have

$$\mathrm{ML}(a, a+n, 2a+n) = n \cdot \mathrm{ML}\left(\frac{a}{n}, \frac{a}{n}+1, \frac{2a}{n}+1\right)$$

By Theorem 5, this becomes

$$n \cdot \mathrm{ML}\left(\frac{a}{n}, \frac{a}{n}+1, \frac{2a}{n}+1\right) = n \cdot \left(3 \cdot \frac{a}{n}\right) = 3a$$

□

Theorem 13.

$$\mathrm{ML}(a, a+3, 2a+3) = \begin{cases} \infty & : a \not\equiv 0 \mod 3 \\ 3a & : a \equiv 0 \mod 3 \end{cases}$$

Proof. If $a \equiv 0 \mod 3$, Theorem 12 gives us $\mathrm{ML}(a, a+3, 2a+3) = 3a$. If $a \not\equiv 0 \mod 3$, then $\gcd(a, a+3, 2a+3) = 1$ and $a \equiv a+3 \mod 3$. Then, Theorem 1 gives us $\chi(\mathbb{Z}, \{a, a+3, 2a+3\}) = 3$, whence $\mathrm{ML}(a, a+3, 2a+3) = \infty$ when $a \not\equiv 0 \mod 3$.

□

3 A Computational Method

In this section, we present a computational method by which the author found $\mathrm{ML}(a, b, c)$ for various a, b, c and were able to observe the patterns that suggested the above formulae. The method was inspired by the logical nature of various brute force attempts made towards 3-coloring the integers with certain forbidden distances.

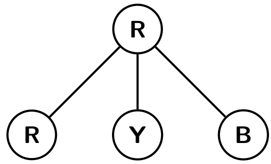
Imagine we are creating a colored ternary tree level by level. The number of a level is the number we are trying to color. We start with Level 1, the root, which we will color red for no particular reason. We start by giving the root three children - one red, one yellow, and one green. Now we repeat the following two steps:

- Check each node on the current level; if any node is the same color as its ancestor a, b, or c levels up, delete that node.

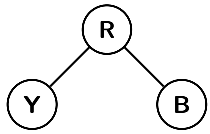
- If any nodes remain, give each of them 3 children (one red, one yellow, one blue). Move on to the next level.

This process will continue for infinitely many levels if and only if it is the case that $\mathrm{ML}(a, b, c) = \infty$. Otherwise, it will terminate when the first step results in the deletion of all of the nodes on a given level. Then, the number of levels in the final tree after the process has ended is exactly $\mathrm{ML}(a, b, c)$. We will show this process in action for $a = 1$, $b = 2$, and $c = 3$ (note that, by Example 1, we already know that $\mathrm{ML}(1, 2, 3) = 3$).

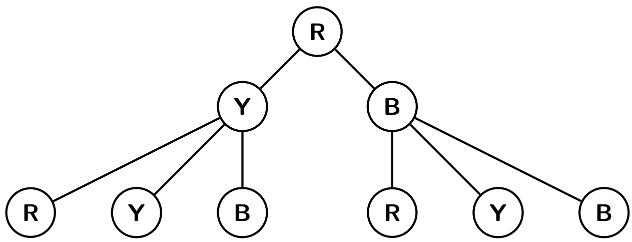
- Start:



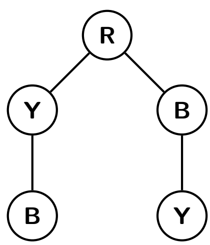
- On Level 2, the red node has a red ancestor $a = 1$ levels above it. Delete the red node on Level 2.



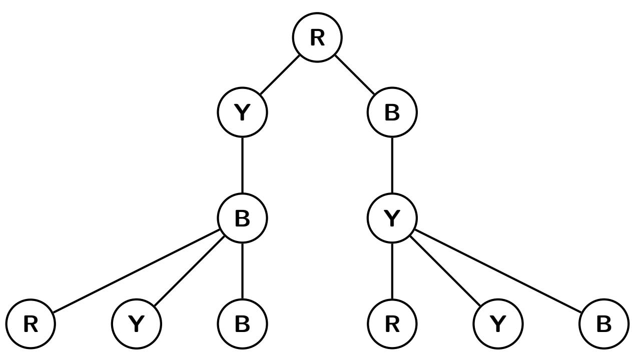
- The other nodes are fine, so we move to Level 3



- Delete each node which is the same color as an ancestor $a = 1$ or $b = 2$ levels above.



- Move on to Level 4



	$a=1$	2	3	4	5	6	7	8	9	10
$b=1$	∞	3	6	∞	9	12	∞	15	18	∞
2	3	∞	6	6	∞	12	15	∞	18	18
3	6	6	∞	9	14	9	17	19	18	24
4	∞	6	9	∞	12	12	∞	12	24	∞
5	9	∞	14	12	∞	15	22	∞	25	15
6	12	12	9	12	15	∞	18	18	18	28
7	∞	15	17	∞	22	18	∞	21	30	∞
8	15	∞	19	12	∞	18	21	∞	24	24
9	18	18	18	24	25	18	30	24	∞	27
10	∞	18	24	∞	15	28	∞	24	27	∞

Figure 1: $\mathrm{ML}(a, b, a+b)$ for $1 \leq a, b \leq 10$

- All of the nodes on Level 4 get deleted for being $a = 1$, $b = 2$, or $c = 3$ levels below an ancestor of the same color. Thus, we stop, and the number of levels of the final tree gives us $\mathrm{ML}(1, 2, 3) = 3$.

In fact, this process actually gave us two 3-colorings of S_3 ($\{R, Y, B\}$ and $\{R, B, Y\}$); although in this case the 3-colorings are the same up to renaming of colors, it is still true that the algorithm will produce *all* 3-colorings of $S_{\mathrm{ML}(a,b,c)}$. This algorithm is deterministic, and will calculate the exact value of $\mathrm{ML}(a, b, c)$ when it is finite. The author has implemented this algorithm in C++, and some results are displayed in Figure 1.
Many patterns can be seen in this table; the following conjecture is based on patterns observed in this data and more.

Conjecture 14.

$$\mathrm{ML}(a, a+4, 2a+4) = \begin{cases} 4a+5 & : a \equiv 1 \mod 2 \\ 4a+4 & : a \equiv 2 \mod 4 \\ 3a & : a \equiv 0 \mod 4 \end{cases}$$

The proof for when $a \equiv 2 \mod 4$ involves a simple application of Corollary 8 and Theorem 11. When $a \equiv 0 \mod 4$, we know the formula from Theorem 12. The conjecture resides in the case $a \equiv 1 \mod 2$.
Another potential area of discussion: given a, b, c forbidden distances, is there a meaningful way to analyze the number of non-isomorphic colorings of sets S_n for $1 \leq n \leq \mathrm{ML}(a, b, c)$? The author has discovered that, for some forbidden distances a, b, c, there is only one coloring (up to renaming of colors) of S_n where $n = \mathrm{ML}(a, b, c)$. We omit any proof of this, but such a proof would be similar to those above that involve seeing what is forced once you assume the colors of two of the numbers, without loss of generality. For instance, the proper colorings of S_n in Theorem 4, when $b \not\equiv 1 \mod 3$, are unique, and something similar holds in Theorem 5.

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References

[1] Jer-Jeong Chen, Gerard J. Chang, and Kuo-Ching Huang. Integral Distance Graphs. *Journal of Graph Theory*, 25(4): 287-294, August 1997.